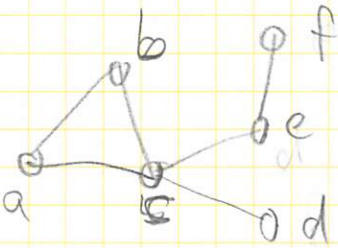
4 Dimension of incidence orders

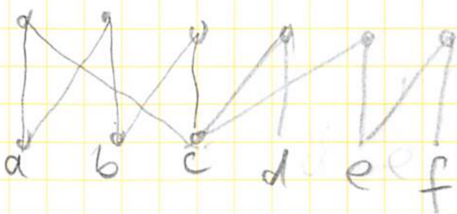
4.1 Introduction

Recall the definition

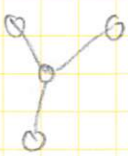
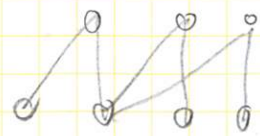
$G=(V,E)$ a graph the incidence order is $P_G=(V\cup E,<)$ with relations $v<e \iff v\in e$

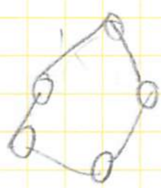
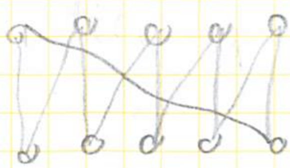
Ex

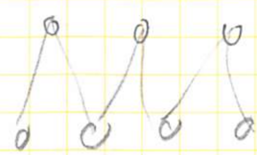




We know

G  P_G  $\dim P_G = 3$

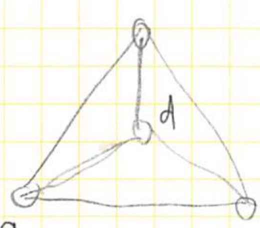
C_k  P_G  $\dim P_{C_k} = 3$

P_k   $\dim P_k = 2$

Cor: $\dim P_G = 2 \iff G$ is a forest of Paths. and $n>1$

$G=K_n$ $\quad \dim(P_{K_n}) \geq \log\log n + 1$

K_4



degree 3 $\Rightarrow \dim P_{K_4} \geq 3$

a 3 realizer

b c d a

a c d b

a b d c

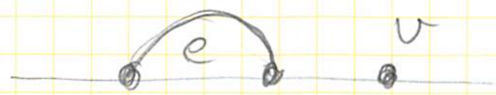
When dealing with $\dim P_{K_n}$ we have only focussed on vertex orderings

Infact we have implicitly worked with this

Def: G Graph $\dim(G) = \min t$ such that

$\exists\, \pi_1 \dots \pi_t \in S_V$ such that $\forall (v,e)$ with $v \notin e$

$\exists i$

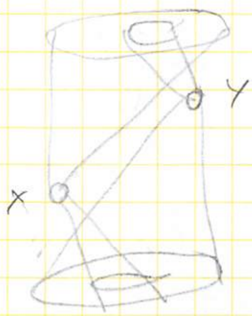


Proposition:

- If G has no leaf vertices
  $\Rightarrow \dim G = \dim P_G$
- If G^+ is obtained from G by adding any number of leaf vertics to each vertex of G

$$\dim G \leq \dim P_{G^+} \leq \dim G + 1$$

proof: • consider critical pairs

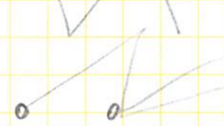


$\dim(G)$ takes care of min-max

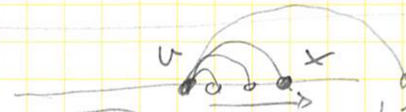
both max

no

both min

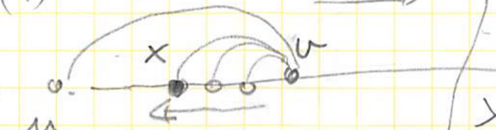


• At a vertex: (1)



u may not be enough

(2)



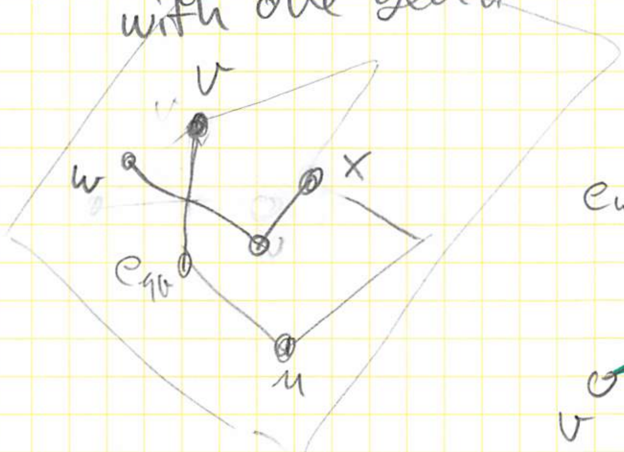
x never gets over vu

Do (1) in each π_i and add one corresp to π_1 in manner (2)

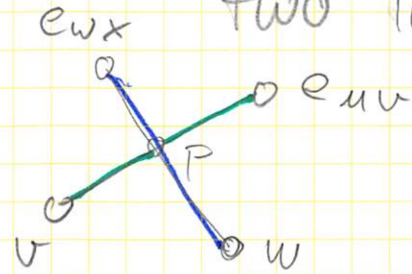
Proposition: $\dim P_G \leq 3 \Rightarrow G$ planar

proof. Let $\dim P_G \leq 3$ and suppose G non plan.
Consider an order preserving embedding of P_G in $\mathbb{R}^3$. Let H be a plane with $(1,1,1) \in H^\perp$ project the elements and comparabilities orthogonally to H

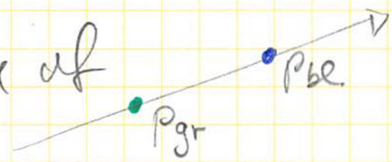
straight line drawing with one bend



non-planar
$\Rightarrow \exists$ crossing of two independent edges.



on the proj line of p



$\Rightarrow$ v to p_{green}
p_{green} to p_{blue}
p_{blue} to e_{wx}
each increasing

$\Rightarrow v < e_{wx}$ ↯ □

In 1989 Schnyder proved the following characterization of planarity

Thm Schnyder G planar $\Leftrightarrow \dim P_G \leq 3$

In this and the next lecture we will discuss this result and some generalizations.

4.2 Schnyder woods

Dimension is monotone
$\Rightarrow$ H a subgraph of G $\Rightarrow$ $\dim H \leq \dim G$

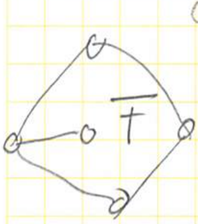
Remark: Every planar graph is a (induced) subgraph of a planar triangulation

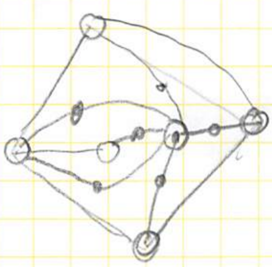
triang are simple graphs!

Proof Fix a plane drawing of G

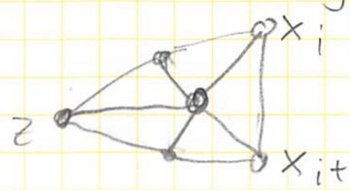
(1) $\forall F$ face G let $x_1 x_2 \ldots x_k$ be the cyclic order of vertices on boundary repetitions allowed

add edges $x_i - y_i^F - z^F$





(2) Triangulate each face of this graph with a new vertex

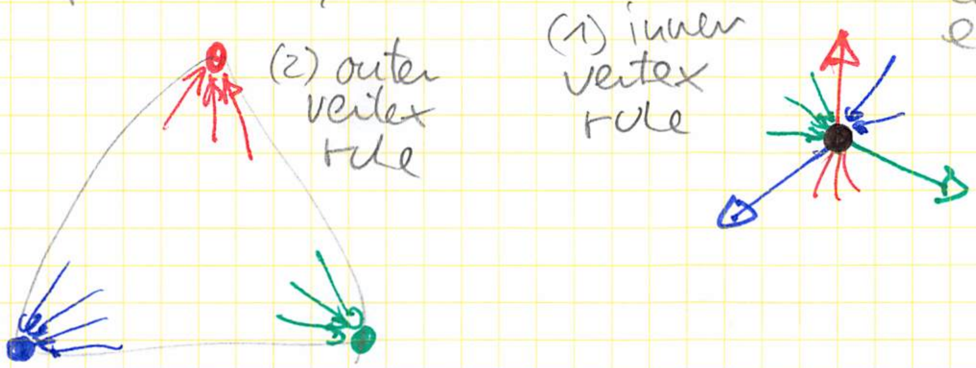


Thm Schnyder G plane triangulation $\Rightarrow \dim G \leq 3$

Walter Schnyder 89/90

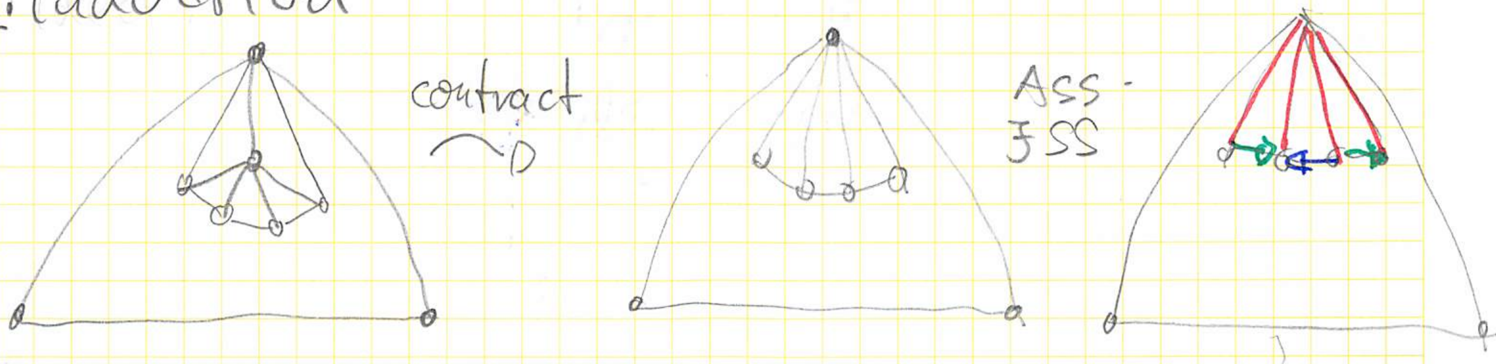
Def. Schnyder wood

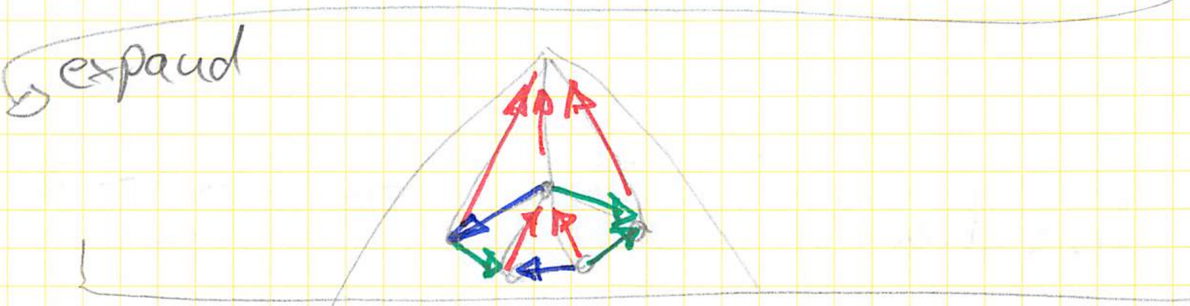
orientation and coloring of inner edges of triangul.



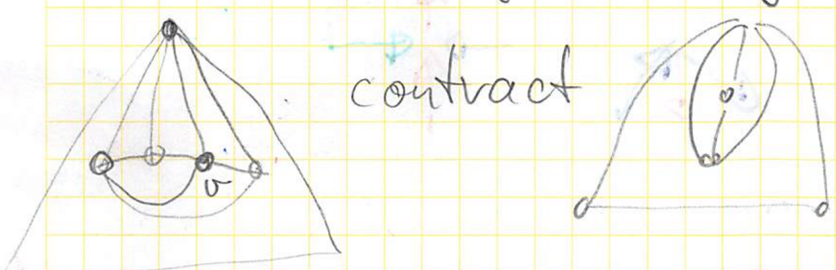
Proposition
Existence of Schnyder woods

Pr: Induction



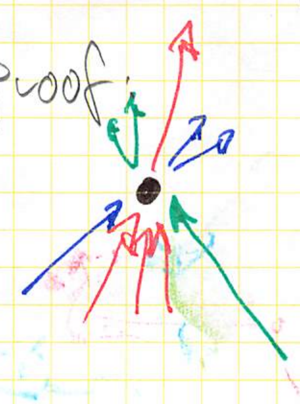


Problem: want to contract $v\,a$
but $\exists$ separating triangle $u\,v\,a$

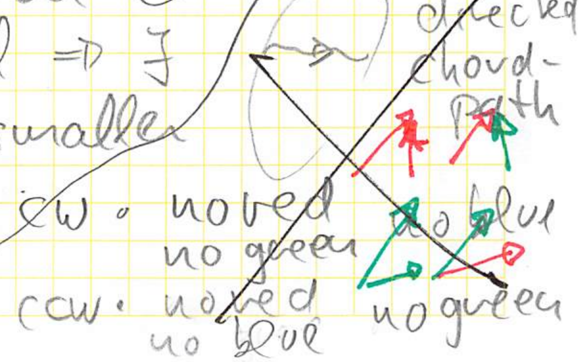


try a neighbor of a inside the triangl $u\,v\,a$ → reduce area of triang

Lemma: Let S be a SW and $\{r,g,b\} = \{1\,2\,3\}$ and let T_i be the set of oriented edges of color i and T_i^{-1} the set with reverse or.
$\Rightarrow$ $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic. → Rückseite

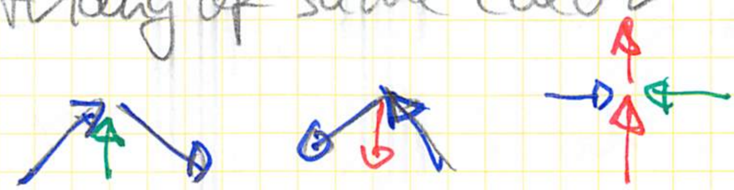
Proof: 

- suppose there is a cycle C
- if C is not facial $\Rightarrow \exists$ directed chord-path
$\Rightarrow \exists$ dir. cycle with smaller enclosed area
$\Rightarrow C$ can be assumed to be a triangle.



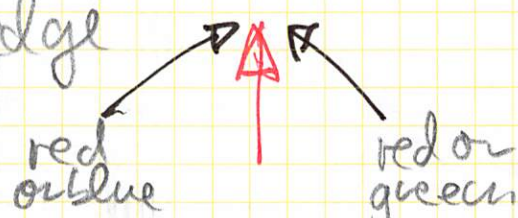
no directed triangle in $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$

obs 1 no two consec edges at triang of same color



$\Longrightarrow$ all 3 colors

obs 2 no red edge



T_i is acyclic $n-3$ edges on $n-2$ vertices

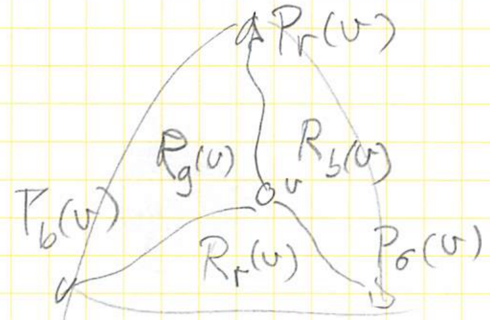
Cor 1. $\Rightarrow$ T_i is a tree
all paths on T_i directed to a_i (center)

Def $P_i(x)$ the path $x \to a_i$ in T_i
(path of x in color i)

Cor 2. $P_i(x) \cap P_j(x) = \{x\}$ if $i \neq j$

Pr. otherwise a cycle in $T_i \cup T_j^{-1}$

Def (region of x in color i)

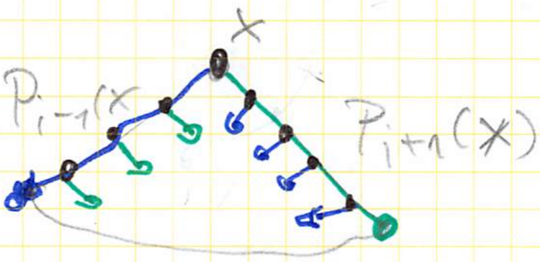


$R_i(v)$ bounded region
with $\partial R_i(v) = P_{i-1}(v) \cup P_{i+1}(v)$
$\cup \{a_{i-1} a_{i+1}\}$

Lemma (containment of regions)

$y \in R_i(x) \quad \underset{y \neq x}{\Rightarrow} R_i(y) \subset R_i(x)$

Proof



Consider local conditions for vertices on paths $P_{i-1}(x)$ and $P_{i+1}(x)$

$\Rightarrow \forall y \in R_i(x) \quad P_{i-1}(y) \subseteq R_i(x)$
and $P_{i+1}(y) \subseteq R_i(x)$

$\Rightarrow R_i(y) \subseteq R_i(x)$ but $x \notin R_i(y)$

$\square$

proof of Schnyder's Thm

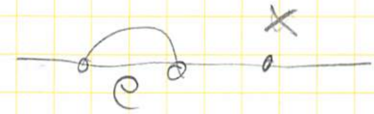
G triang   S a SW of G $R_i(x)$ region

Let $\pi_i : V \to \mathbb{R}$ be such that

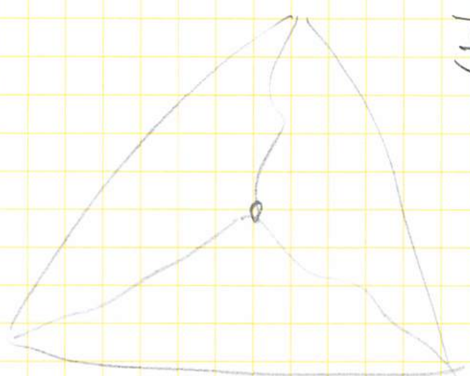
$R_i(x) \subset R_i(y) \Rightarrow \pi_i(x) < \pi_i(y)$

(even though π may have ties think of it as a linext of the inclusion of regions)

Claim $\pi_r\ \pi_g\ \pi_b$ certify $\dim(G) \leq 3$

Have to show $\exists i$  $\forall$ pairs (e,x) with $x \notin e$

Let (e,x) be a pair with $x \notin e$



$\exists i$ such that $e \subset R_i(x)$

$\Rightarrow$ both vertices of e belong to $R_i(x)$

$\Rightarrow$ both below x in π_i

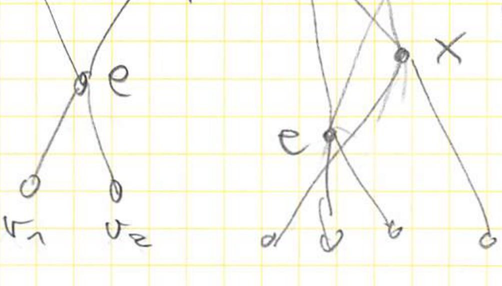
summary □

Two | Extensions and Applications |

(I) Consider the incidence poset of vertices-edges and bounded faces $P_{VEF'}(G)$ of a triangulation

Proposition: $\dim P_{VEF'}(G) \leq 3$

Claim: $\forall e \in E$ e is not contained in a crit pair



(e,x) $up(x) \subseteq \{f_1 f_2\}$

$\Rightarrow$ x is a face. But a face containing both vertices of e contains e

triang. are simple!

$(x\ e)$ $down(x) \subseteq \{v_1 v_2\} \Rightarrow$ x is a vertex incident to $f_1 f_2$

- $\Pi_r\ \Pi_g\ \Pi_b$ revert all pairs (v, f) with $v \in f$

Proof f is contained in one of the regions of v. $\square$

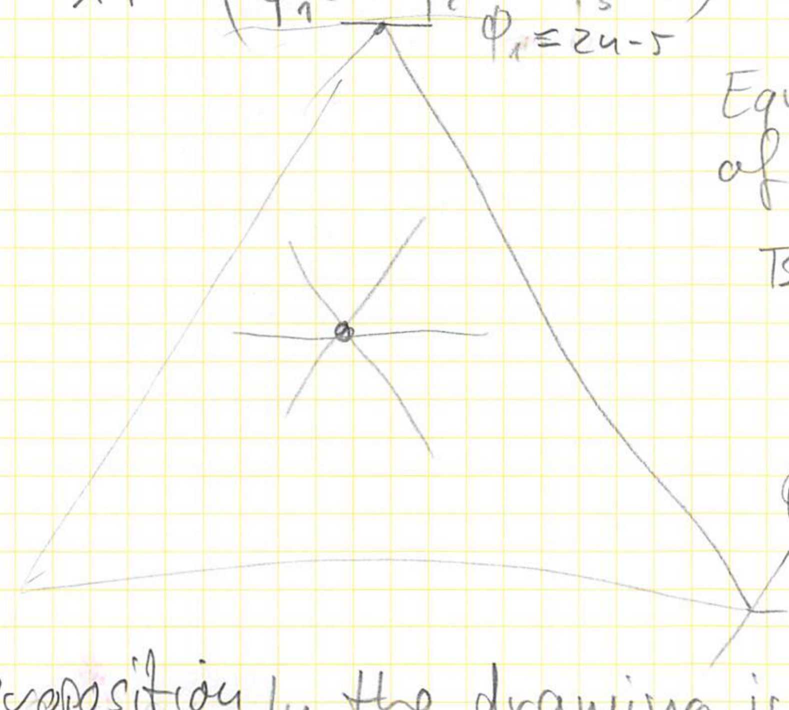
Cor $\dim P_{VEF}(G) \le 4$
here we include the outer face

∘ We know $\dim(P - x) \le \dim(P) + 1$

II Planar drawings

Let $\phi_i(x) = |R_i(x)|$ we get a mapping

$x \mapsto (\phi_1(x)\ \phi_2(x)\ \phi_3(x))$ with $\sum \phi_i(x) = 2n-5\ \forall x$

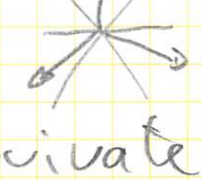


Equilateral triangle of height $2n-5$

Barycentric coordinates unique point $\forall x \in V_G$

Proposition In the drawing induced by ϕ

i. each i-edge is in the i-wedge

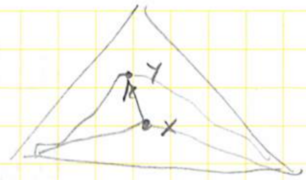


ii. each edge has its private triangle empty

~~iii. the rotation at every vertex is preserved~~

(iii) there is no crossing

Proof (i)

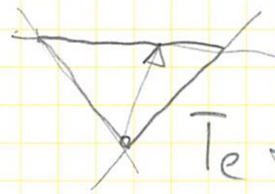


$\forall e$ i-edge

$\phi_i(x) < \phi_i(y)$ and $\phi_j(y) > \phi_j(x)$ for $j \neq i$

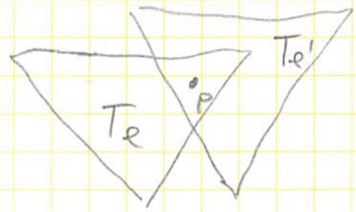
(ii) consider $v \notin e = (x,y)$ $\Rightarrow \exists i$ st $e \subseteq R_i(v)$

$\Rightarrow \forall x \in e: \phi_i(x) < \phi_i(v)$



empty T_e triangle of edge

(iii) suppose that there is crossing pair e, e' of edges.



Look at T_e and $T_{e'}$

cross point $p \in T_e \cap T_{e'}$

$\Rightarrow$ vertex of e in $T_{e'}$ (or conversly) □

Cor: A planar n vertex graph G has a straight line drawing with vertices on grid points in $[2n-5] \times [2n-5]$

III Triangle contact representations

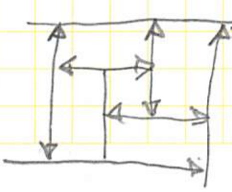
Intersection representations of graphs

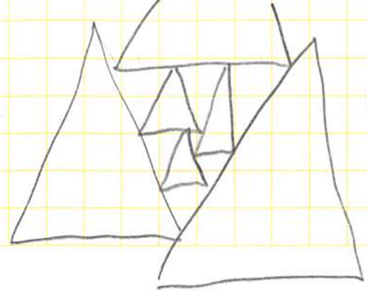
- interval graphs
- rectangle intersection
- disc graphs

Contact representations of planar graphs

- ▷ v-objects are internally disjoint
  (v ≙ vertex)
- ▷ contact $\Leftrightarrow$ edge

segment contact of cube





triangle contact of octahedron

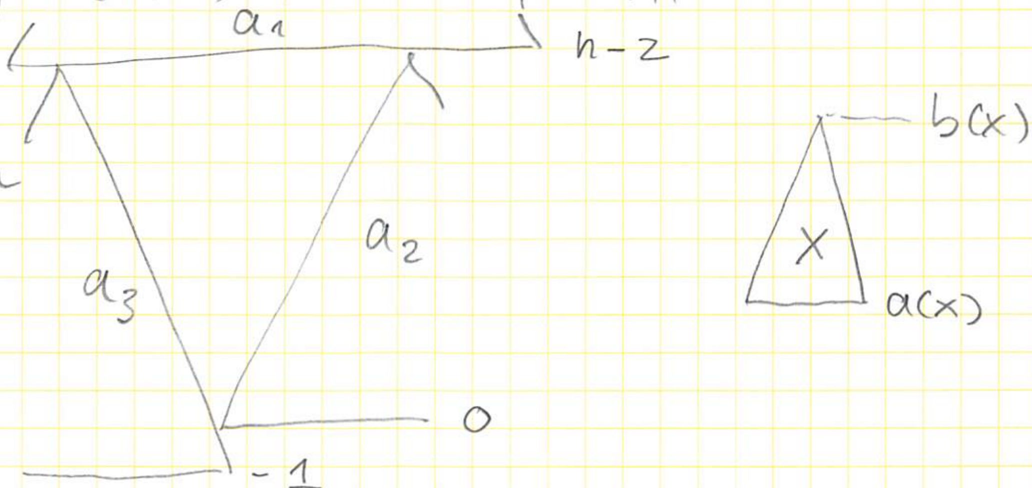
THM Every planar graph G has a triangle contact representation

Proof: $\exists\, T$ triangulation containing T as induced subgraph.

Consider SW of T. Let π be a top ordering of $T_1 \cup T_2^{-1} \cup T_3^{-1}$ here $\pi_1 \dots \pi_{n-3}$ list of inner vertices. With x we consider $x \xrightarrow{1} y$ and let

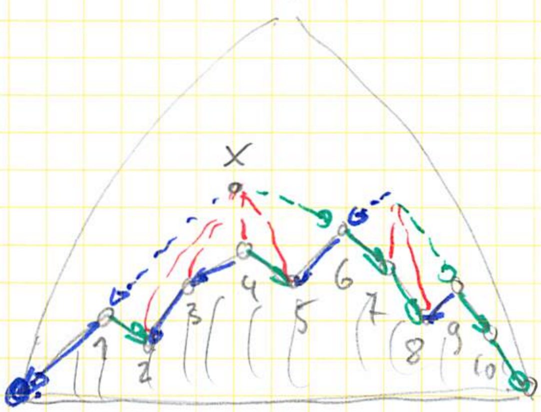
$a(x) = pos(x,\pi) \wedge b(x) = pos(y,\pi)$

The repr.

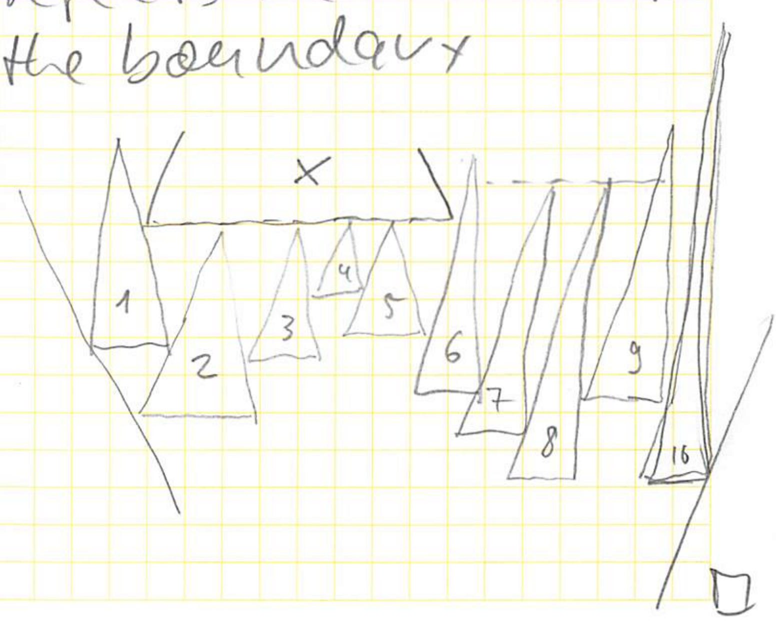


The repr. is constructed along the ordering π

Obs: $\forall k \in [1, n-3]$ $\pi_1 \dots \pi_k$ consists of an initial part of the SW

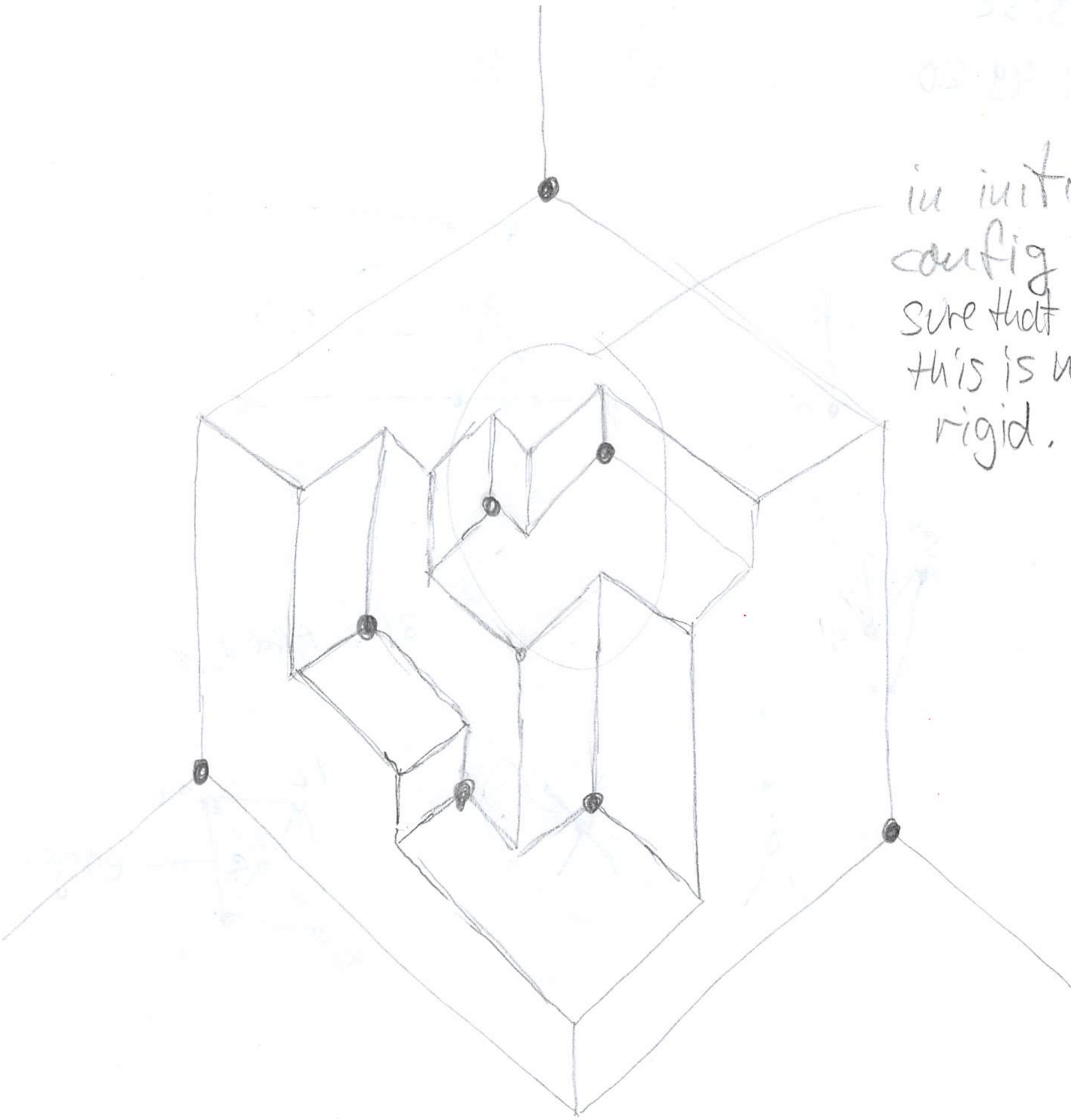


Invariant: the order of unclosed triangles reflects the order of the boundary



□

Top.

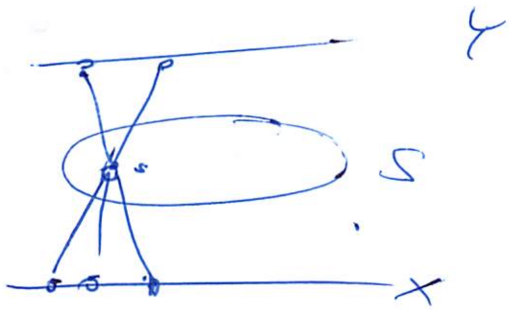


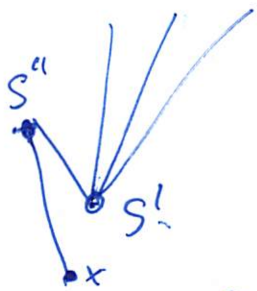
in initial config make sure that this is not rigid.

10:30 115B

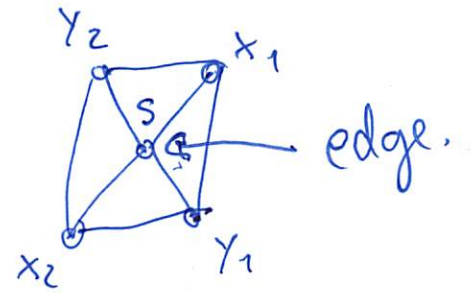
3:32

1:49:20





$3|S| + |X| + |Y|$



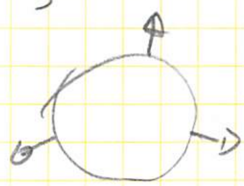
Have seen $\underline{\underline{SW}}$

dim $P_{VEF'} \le 3$
grid drawing

OS triangle contact repr.

4.3 3-connected planar graphs and orthogonal surfaces

We consider 3-connected planar graphs with outer vertices $a_1\, a_2\, a_3$ clockwise and half edges at the a_i

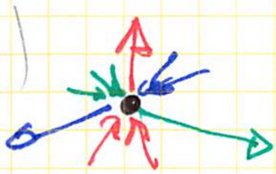


For short this is a Σ-graph.

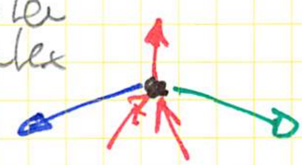
Def: A Schnyder wood of a Σ-graph is a coloring and orientation such that

(i) (edge cond) $\circ \xrightarrow{i} \circ$ uni- $\circ \xrightarrow{i} \circ \xleftarrow{j} \circ$ bidir $i \neq j$

(ii) (inner vertex)



(iii) outer vertex



(iv) no monochromatic dir face cycle

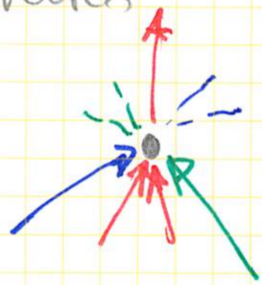
Prop: Every Σ-graph has a SW
proof omitted – no nice proof known

Lemma: S a SW T_i edges of col i
$\Rightarrow T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic (up to bidirected paths)

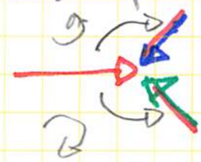
Proof Only care of cycles with area
min area $\Rightarrow$ no internal edge $\Rightarrow$ face

Consider facial cycle C cw or ccw

A vertex



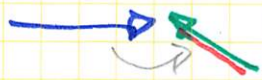
Red
- no unidirected red edge on C



- bidir

yields red dircycle ↯ to (iv)

Blue
- Blue and cw



No

- Blue and ccw

No green cycle (iv)

□

Cor. T_i is a tree arborescence with half-edge at root

Def Paths $P_i(x)$

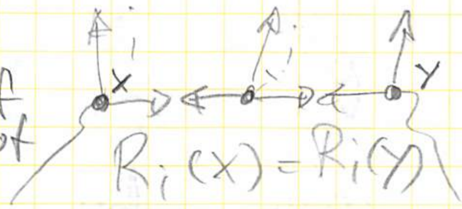
Cor $\forall i \neq j \quad P_i(x) \cap P_j(x) = \{x\}$

Def Regions $R_i(x)$

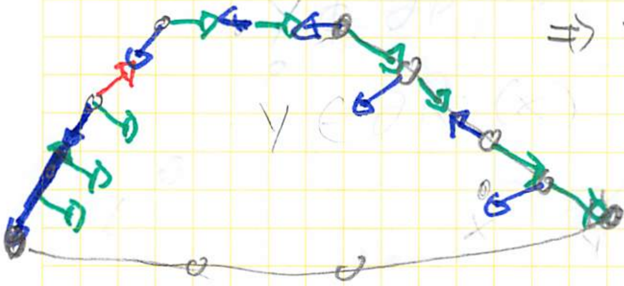
Lemma

$y \in R_i(x)$, $x \neq y$ then $R_i(y) \subsetneq R_i(x)$

if not



Proof Consider local cond. for $y \in P_{i-1}(x) \cup P_{i+1}(x)$



$\Rightarrow \forall y \in R_i(x)$

$P_{i-1}(y)$ and $P_{i+1}(y) \subseteq R_i(x)$

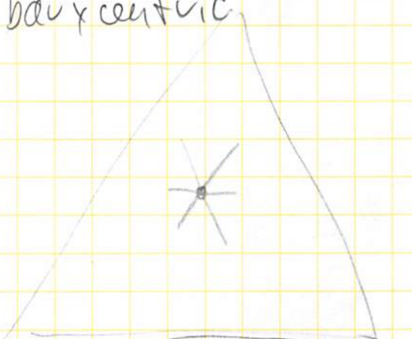
$\Rightarrow R_i(y) \subseteq R_i(x)$

if $x \notin R_i(y) \Rightarrow R_i(y) \subsetneq R_i(x)$ □

define   define f

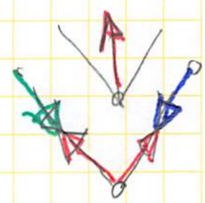
$\boxed{\text{Drawing}}$ $\phi_i(x) = |R_i(x)|$ again $\sum \phi_i(x) = f-1$

barycentric



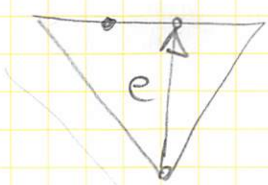
- i-edge is in the wedge

uni-directed interior
bidirected boundary



!! • bidirected path collinear

- weakly empty triangle

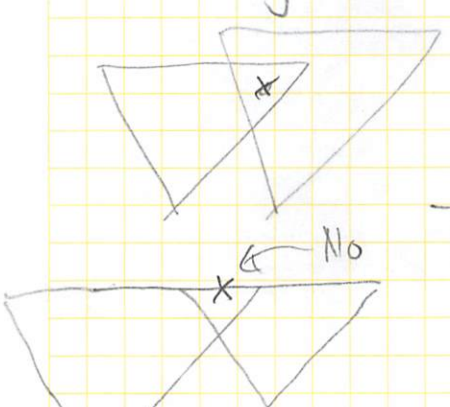


$v \notin e$

$\Rightarrow e \in R_i(v)$ for some i

$\Rightarrow \phi_i(x) \leq \phi_i(v) \quad \forall x \in e$

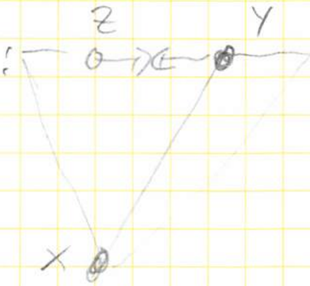
crossing



THM: 3con. pl. gr. has plane straight line convex drawing on $(f-1)\times(f-1)$ grid.

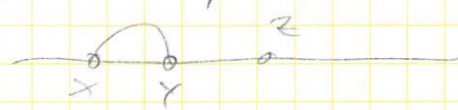
while for triangulations the dimension theoretic applications and the drawing were equally easy. There is a difference in the 3 connected case.

The problem:



we have to bring z above xy

need π_i



We solve the problem using orthogonal surfaces and even prove a strong version of the dimension theorem

Theorem: G Σ-graph $\exists$ OS S such that the critical points of S yield an order embedding of $P_{VEF'}$ in $\mathbb{R}^3$

$\exists$ OS such that crit pairs yield order emb. of P_{VEF} in $\mathbb{R}^3$

~~extend ... and ... we get embedding ... $f(y)$ ... $\{f(x) : x \in F\}$~~

in particular $\dim P_{VEF'} = 3$

Orthogonal surfaces

$V \subseteq \mathbb{R}^3$ finite $\quad U[V] = \{x \in \mathbb{R}^3 : \exists v \in V \; v \leq x\}$

$S_V = \partial U[V]$

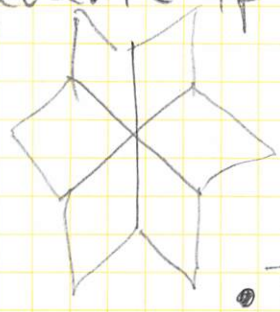
the orthogonal surface generated by V

Note $S_V = S_{\mathrm{Min}\,V}$ $\qquad$ typically V antichain

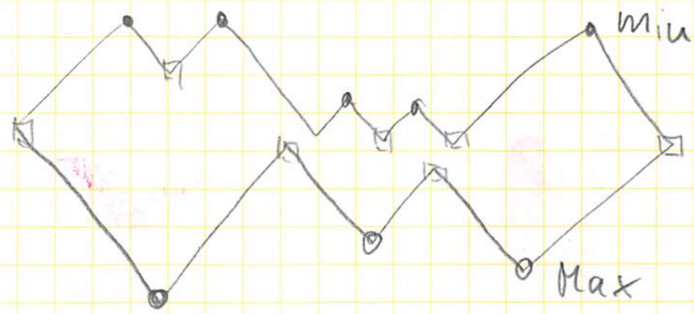
S_V has vertices edges faces for distinction we call them crit points arcs flats

- every arc is incident to two flats
- crit points are minimal / maximal / saddle

S_V is generic if every saddle is incident to 3 flats



- generic $\Rightarrow$ every crit point is incident to three flats
- The typical shape of a flat of type 1 coord. 1 is constant



M a plane graph. A drawing $M \hookrightarrow S_V$ is a geodesic embedding if

(G1) (vertex ax) Bijection $V_M \leftrightarrow V$

(G2) (edge ax) every edge (u,v) is geodesic i.e. a union of two segments connecting u and v to $u \vee v$ and every arc incident to v is part of an edge

(G3) ("planarity") there is no crossing of edges

(show a geodesic emb on the example)

Def: S_v is axial if it has 3 orth rays

Prop: S_v generic + axial $M \hookrightarrow S_v$ geodesic
$\Rightarrow$ S_v induces a Schnyder wood on M
(show on drawing – have to check prop)

Prop: Schnyder wood on Σ-graph M
induces an ortho surf. S with $M \hookrightarrow S$

Proof: Schnyder wood $x \mapsto \phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))$
The grid drawing has the empty edge and empty face property $\Rightarrow$ the surface generated by $\mathcal{V} = \{\phi(v) : v \in V_M\}$ has the property. $\square$

Rem: The surface may allow $M' \hookrightarrow S$ with $M' \neq M$

Def: An OS S is rigid if $\forall$ crit points $u \vee v \in S$
$u \vee v > w \Rightarrow w = u$ or $w = v$

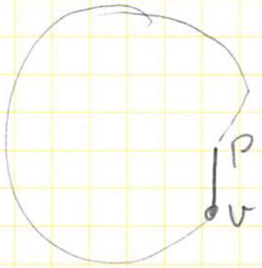
Note: S rigid $\Rightarrow \exists$ unique $M = (V, E)$ with $M \hookrightarrow S$ geodesic. Moreover, positions of critical points yield $P_{VE} \hookrightarrow \mathbb{R}^3$ order preserv.

Prop: S rigid $M \hookrightarrow S$ geodesic f bounded face of M
Let $\alpha_f = \bigvee \{v ; v \in f\}$ then

1. $\alpha_f \in S$ 2. $w \in \mathcal{V}$, $w < \alpha_f \Rightarrow w \in f$

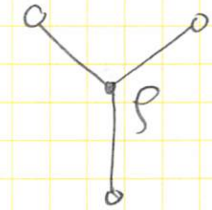
Proof: (1) Let C be the boundary cycle of f

$e \in C \Rightarrow e=(u,v)$ contains an arc (vp), $p = u \vee v$



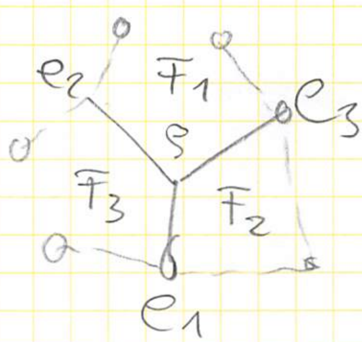
Let F be the flat incident to (vp) wich is (partially) inside f

$\exists$ max point ρ on F "above" v
$\Rightarrow \rho$ is inside f. ρ has 3 arcs each ending in a saddle

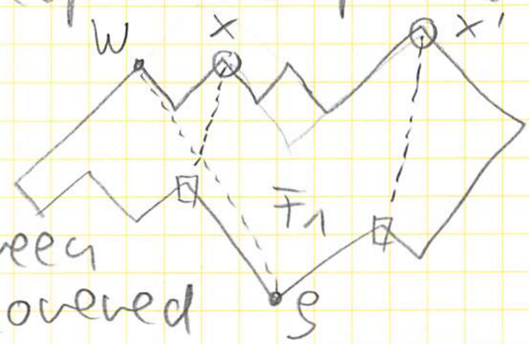


each of these saddles is used by an edge of M

The vertices are on the respectie flats



shape of F_1



the $\wedge\wedge$ between x and x' is covered by edges of M

$\Rightarrow$ all vertices of f are on flats F_i

$\Rightarrow \rho \geq x$ for all $x \in f$ but $\rho = e_1 \vee e_2 \vee e_3$

$e_i = u \vee v$ with $(u,v) \in C \Rightarrow \alpha_f \in S$

---

(2) $\forall w < \alpha_f \quad w \in F_i$ for some i

suppose $w \in f \Rightarrow$ (see the sketch up there)

$\Rightarrow$ the segment (α_f, w) intersects $(e_j x)$

$\Rightarrow e_j > w$ ↯ e_j is only above its two vertices which belong to f.

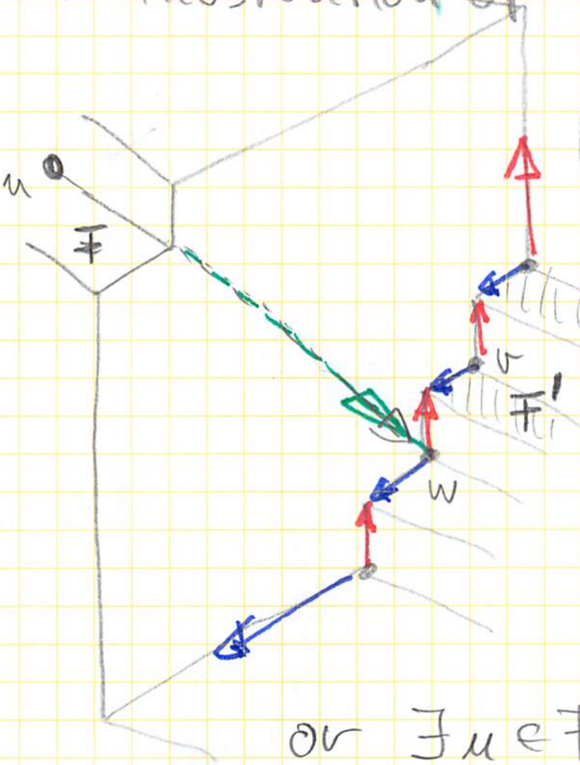
Rigid surface S with $M \hookrightarrow S$ geodesic certifies strong dimension theorem.

THM
$G=(V,E)$ Σ-graph $\Rightarrow \exists\, S$ rigid with $G \hookrightarrow S$

proof. Construct a SW and let S be the OS obtained by counting faces in regions.

An illustration of a problem

We augment $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$



Let $\mathcal{F}_r$ be the set of red flats

we define

$F \to F'$ if

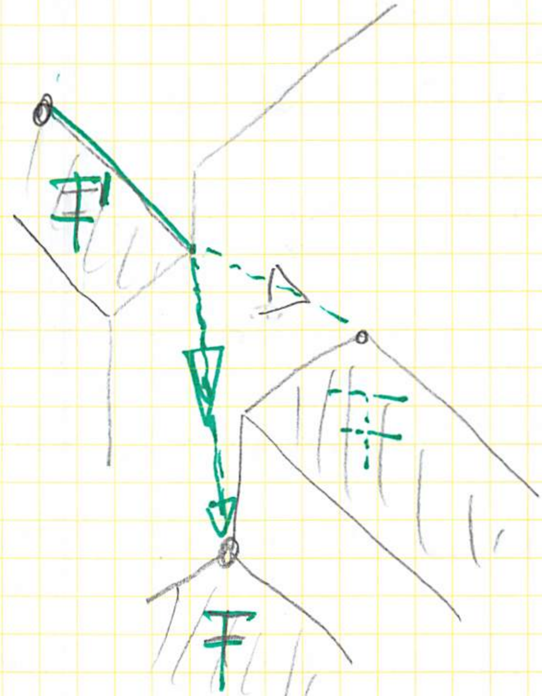
Bilder Rücks.

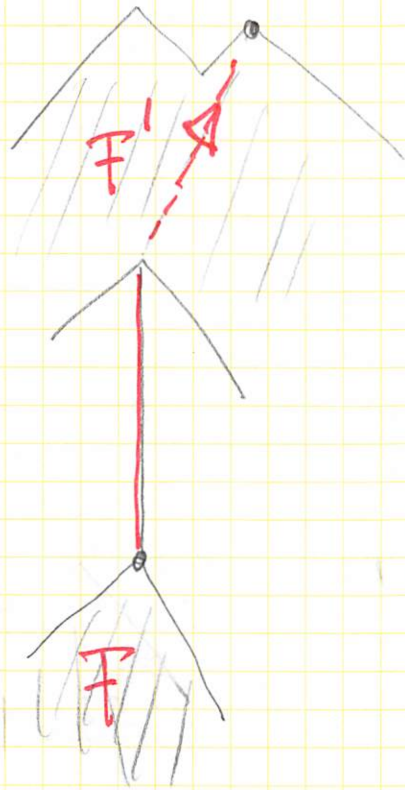
$\exists\, u \in F$ $v \in F'$ with
$(u,v) \in T_i$ or
$(v,u) \in T_{i-1}$ or $(v,u) \in T_{i+1}$

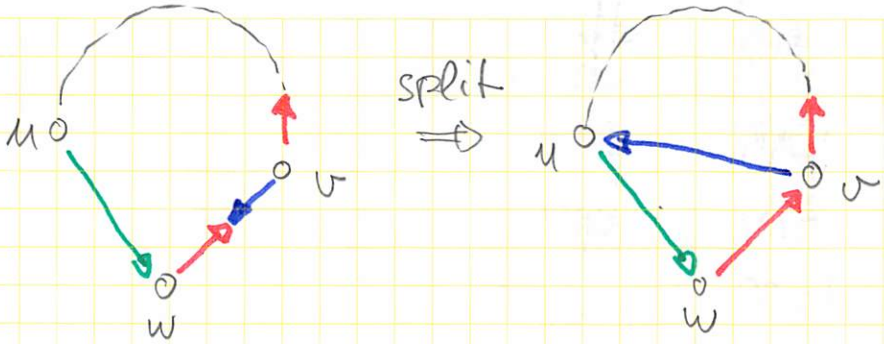
or $\exists\, u \in F$, $v \in F'$ and w
such that $uw \in T_{g+1}$ u,v on common flat of $\mathcal{F}_{g+1}$
and $w \leadsto v$ in T_r
or same with b replacing g

Claim: This relation is acyclic

Proof: We modify the underlying SW such that the relation becomes $T_i \cup T_{i-1}'^{-1} \cup T_{i+1}'^{-1}$


F'
F'
F


F'
F



valid SW

$u \to v$ in T_b^{-1}

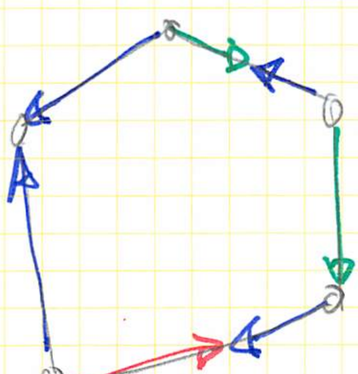
Using $F_i \to \mathbb{R}$ respecting the relation we obtain a rigid OS for G_i.
The skeleton of the surface is unchanged just the heights of flats are changed.

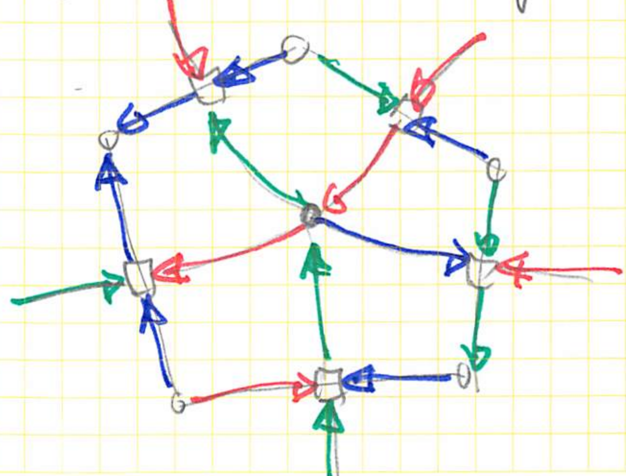
Remarks about SW

- The operation merge

can be used to reduce the number of faces $\longrightarrow$ more compact drawings

- Looking at OS we can see that SW come in primal dual pairs which can be combined in one picture





Dimension of polytopes

G planar 3-con $\Rightarrow \exists$ rigid OS $G \xhookrightarrow[\text{geodesic}]{} S$

$\Rightarrow \dim P_{VEF'}(G) \leq 3$

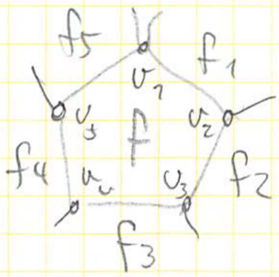
$\Rightarrow \dim P_{VEF} \leq 4$

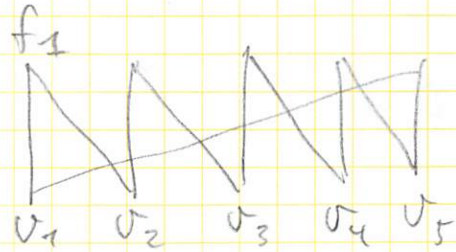
P_{VEF} is the truncated face lattice of a 3-polytope

The Lower bound

<u>THM:</u> G 3 connected $\Rightarrow \dim P_{VF}(G) \geq 4$

proof we know $\dim P_{VF} \geq 3$ because G has a cycle





Suppose $\exists$ realizer $L_1 L_2 L_3$

Let f be lowest face in L_3

$\Rightarrow$ in L_3 $\{v_1 \ldots v_k\} < f < \{f_1 \ldots f_k\}$

$\Rightarrow L_3$ reverses no crit pair of the cycle ↯ $\square$

Remark: In general P a d-polytope

F(P) face lattice $\Rightarrow \dim F(P) \geq d+1$

(essentially the same proof — cycle is a 2-polyt)

But this lower bound can be bad

$\exists$ 4-polytop P with K_n in skeleton

$\Rightarrow \dim F(P) \geq \log\log n$

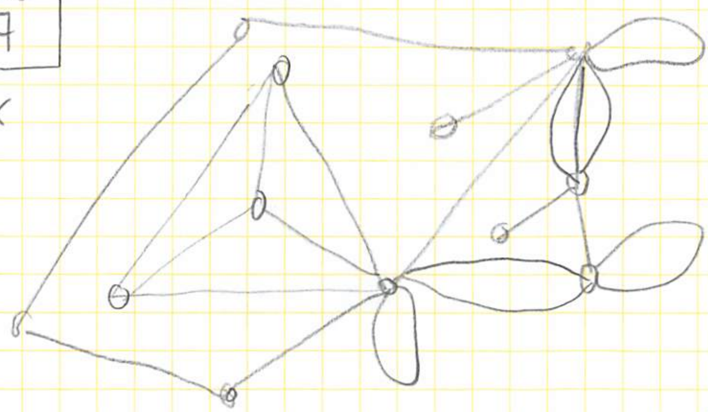
4.4. Dimension of planar maps

in this section we prove the thm

THM [Brightw. Trotter 1997] If G is a plane multigraph, Loops allowed
$\Rightarrow \dim P_{VEF} \leq 4$

The original proof Schnyder tech tour de force. Here a more recent proof which uses several techniques of independent interest

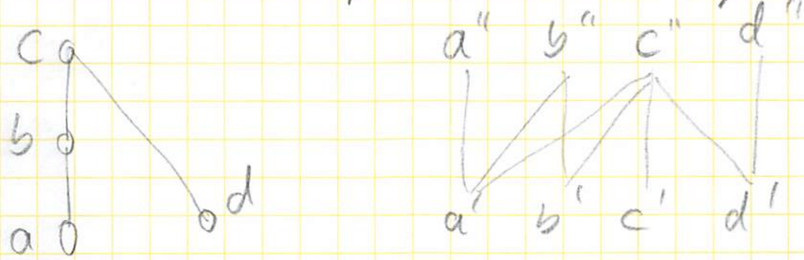
Ex



Splits and dimension

Def $P = (X, \leq)$ a poset the split $sp(P)$:
Ground set 2 copies X', X'' of X
Relation $x' < y'' \Leftrightarrow x \leq y$ in P

Ex



Rem. It is also possible to split only a subset of the elements of P

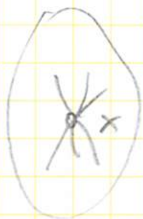
Prop: $\dim(P) \leq \dim sp(P) \leq \dim P + 1$

proof: Let $(x,y) \in \mathrm{Inc}(P)$ be represented by $(x', y'') \in \mathrm{Inc}(sp(P))$

This maps alternating cycles to alt cycles $\Rightarrow \chi(\mathcal{H}_P) \leq \chi(\mathcal{H}_{sp(P)})$

For second ineq.

Let

P 

$\longrightarrow$ $P^{>}$ 

$sp(P)$ is an induced suborder of $P^{>}$

$L_1 \ldots L_k$ a realizer of P

define $L_i^{>}$ by replacing x in L_i by $x'xx''$ consecutive triple

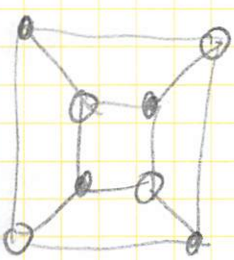
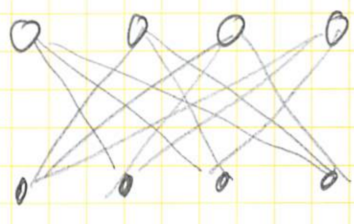
inc pairs that are not reversed by $L_1^{>} \ldots L_k^{>}$

(x'',y'), $(x'y')$ and (x'',y'') with $x<y$

Let $L_{k+1}^{>} = \overleftarrow{L_k[X']}\; L_k[X]\; \overleftarrow{L_k[X'']}$

Dimension of GIGs

Every bipartite graph can be seen as a height 2 poset

Ex  

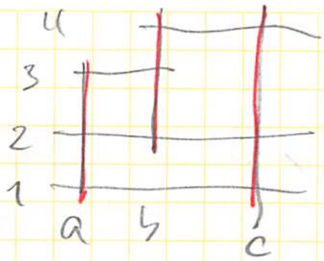
It makes sense to talk about the dimension.

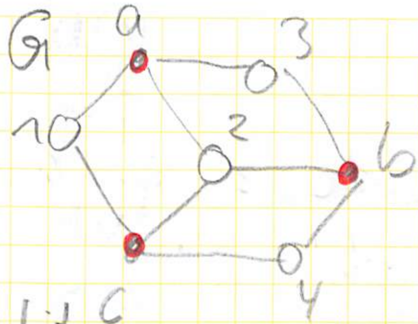
Def A GIG is a grid intersection graph

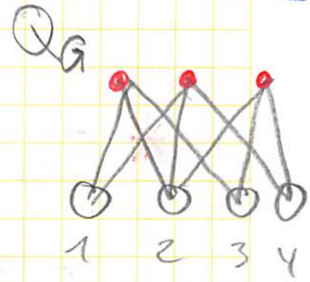
vertices; horizontal and vertical segments with distinct support lines

edges; pairs of intersecting segments

Ex:







obs: GIGs are bipartite

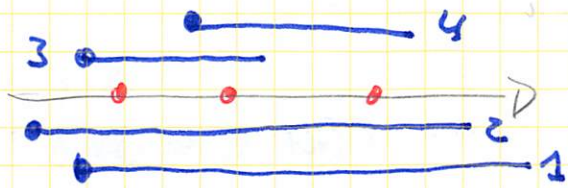
Prop. G a GIG $\Rightarrow$ $\dim Q_G \leq 4$

proof: Construct 4 Lin extensions

$L_E \; L_W \; L_S \; L_N$

L_α in dir α take mins as early as poss
maxes as late as possible

L_E

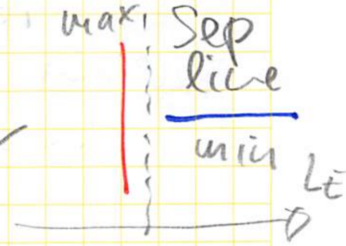


1 2 3 a 4 b c

L_W 1 2 4 c 3 b a

claim: they are a realizer

proof incomp min max pair



$\square$

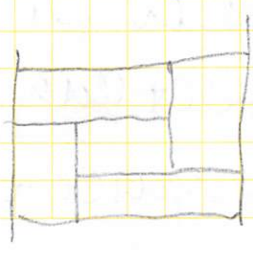
Segment contact representations

Theorem: Every planar bipartite graph admits a segment contact representation with interiorly disjoint horizontal and vertical segments.

* Rückseite

For the G above

just cut ends



the cube

* Ein wunderbarer Beweis für den Satz geht so:

cf Lovasz Buch [gomhep]

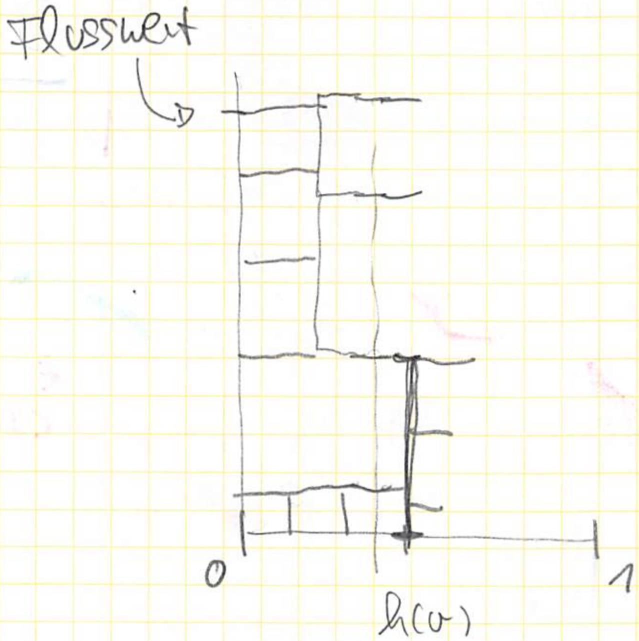
Input: Q quadrang G_s der schwarze Graph von Q mit s, t

$h : V_g \to \mathbb{R}$ harmonisch mit polen $h(s) = 0$ $h(t) = 1$

Sei $f(x,y) = h(y) - h(x)$

das ist ein Fluss

Mit einem sweep von links nach rechts



Konstruktion eines squarings

Vorsicht degeneriertheiten $h(w) = h(v)$

sowie

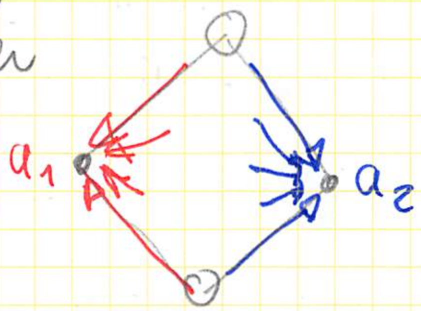
Aber wir wollen ja nur eine rectangulation dh. wir können die Degeneriertheiten durch Perturbation auflösen.

Fact: Every bip. planar graph is induced subgraph of a (planar) quadrangulation

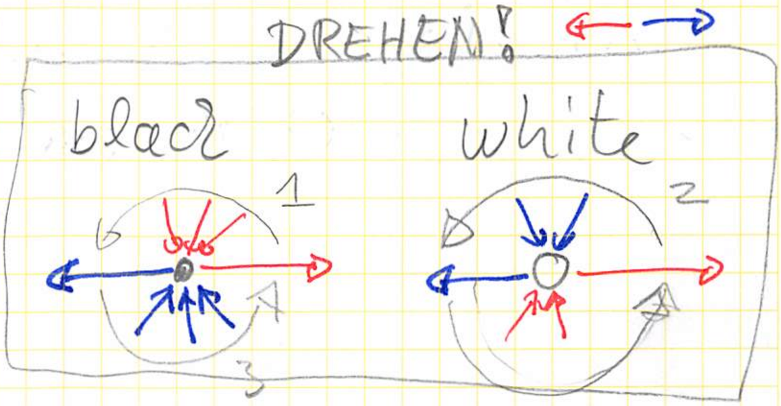
From now on we look at quadrangulations with white and black vertices

Def: A separating decomposition of Q is a coloring and orientation of the edges such that

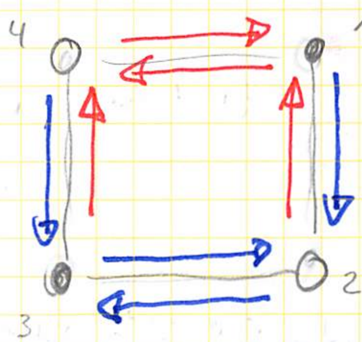
outer



DREHEN!



Claim faces in a SD are as follows (8 possibilities)

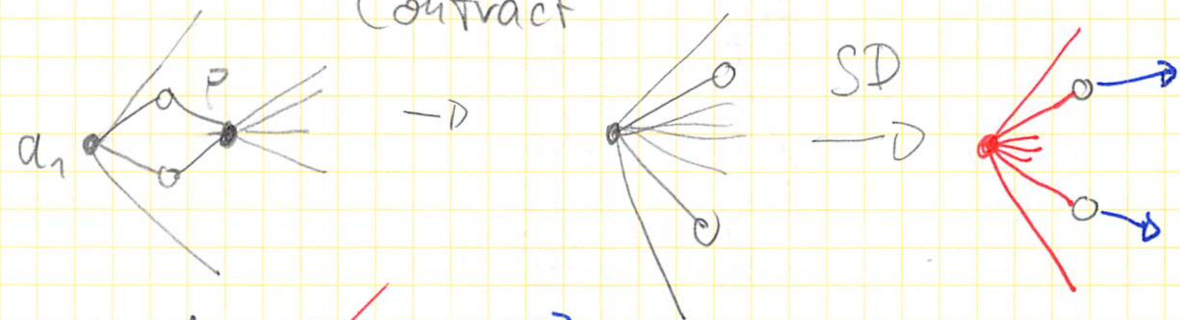


Proposition: Every quadr. Q has a SD

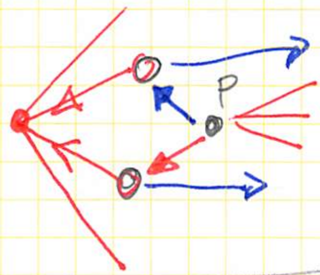
Proof: By induction $n=4$ ✓

- If Q has a vertex of deg 2 remove - induct reinsert (the case that it is a_i is special but easy)
- Now Q has a inner vertex v adjacent to a_1
  Let p be a black neighb. of v $p \neq a_1$

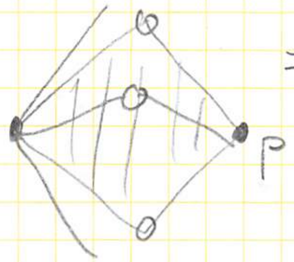
Contract



Expand



if p and a_1 have 3 com neighbors contraction yields double edges.



$\Rightarrow$ no deg 2 $\rightarrow$ there is another candidate p' in the region iterate find p

Lemma: Let T_i be the i-colored edges of a SD $\Rightarrow$ $T_r \cup T_b^{-1}$ is acyclic

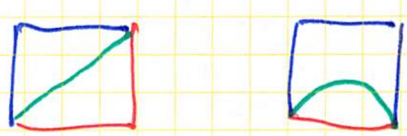
Proof: Suppose there is a cycle chord or inner vertex yield cycle of smaller area, C is a face look at shape no ⚡ □

Cor. T_i is a tree

Obs: Every inner face has exactly two color changes.

*Rückseite

Def the sep arc of a face is a curve separating the colours



Ein alternativer Ansatz um die alternating trees zu bekommen:

Wir definieren $R(x)$ die region von x als den Bereich oberhalb von $P_r(x) \cup P_b(x)$

Beob $\forall x,y$ entweder

- $R(x) \subset R(y)$ od $R(y) \subset R(x)$
- $y \in R^\circ(x) \Rightarrow R(y) \subset R(x)$

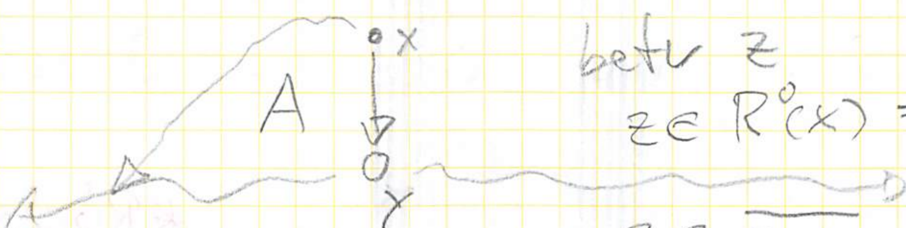
Sei $\varphi(x) = \#$ faces in $R(x)$ sowie

$\varphi(a_r) = -1 \quad \varphi(a_b) = n$

Def: alternating tree

Prop: Die Abb $x \mapsto \varphi(x)$ und die Zeichnung der blauen Kanten als Bögen über der Achse, der roten unter der Achse liefert zwei alternating trees.

Bew: Betr Kante hier $(x\,y)$ blau



betr z

$z \in R^\circ(x) \Rightarrow z$ vor $x\,y$

$z \in \overline{R(y)} \Rightarrow x\,y$ vor z

$z \in A \Rightarrow P_b(z)$ geht durch y (lokale Bed bei x!)

...... weitere Fälle

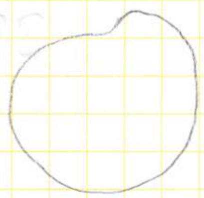
THM 2.14 aus Binary Labelings

Felsner
Huemer
Kappes
Orden

Obs: Every inner is incident to two separating arcs

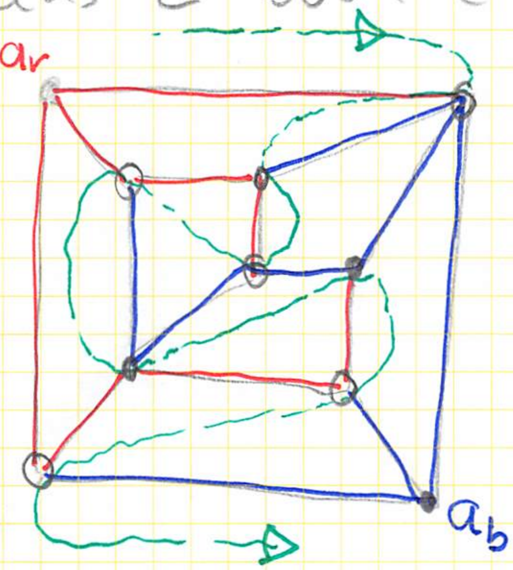
⇒ sep arcs form a curve connecting the two outer white vertices plus possibli closed curves

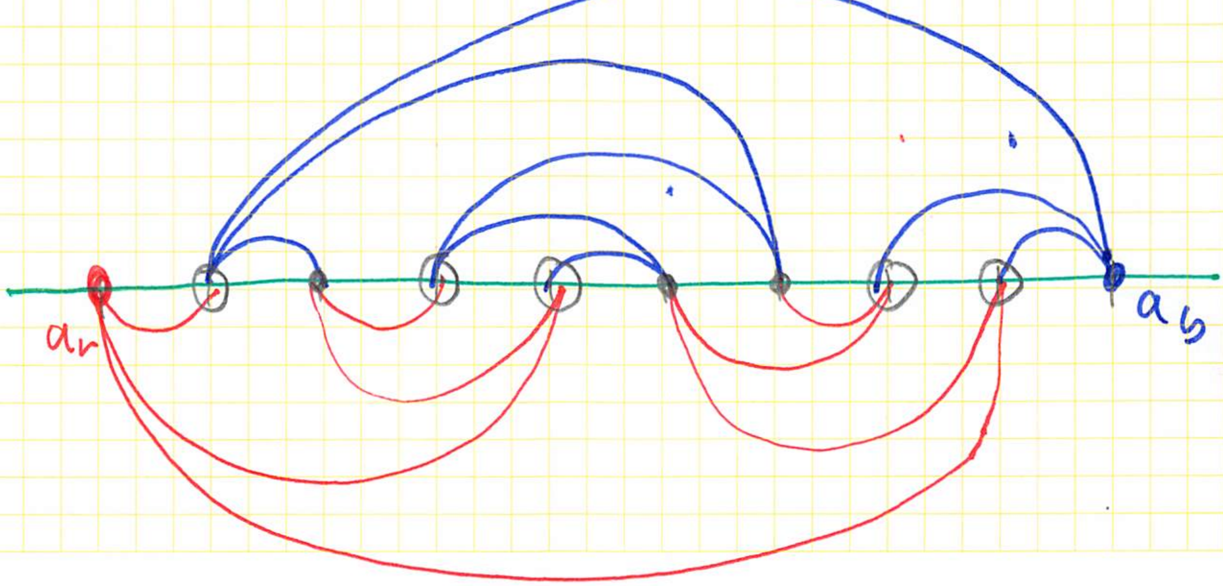
Claim No closed curves

proof  inner edges have paths to root of their tree ↯ separation □

Stretching the separating curve yields 2 book embedding.

Ex

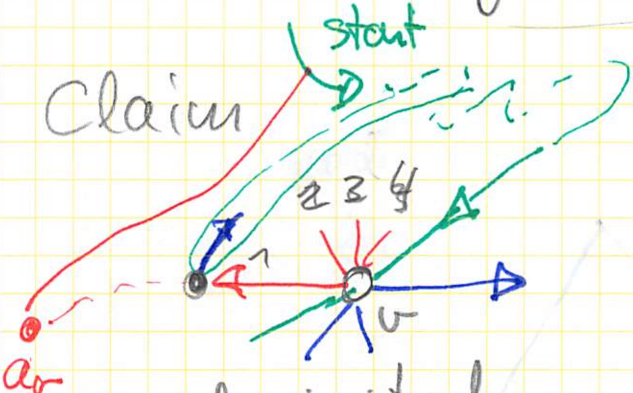


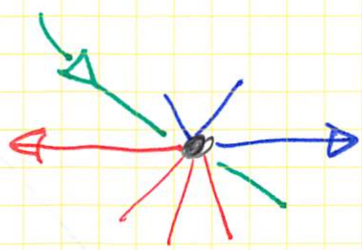


Def: An alternating layout of a plane tree is a drawing in a halfplane with all vertices on the boundary line such that each vertex has all neighbors on one side

Prop: the 2-book layout yields alternating drawings of T_r and T_b

Claim





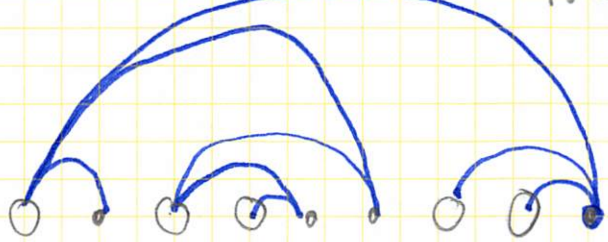
use separation property and cross freeness

red visited before v
blue after

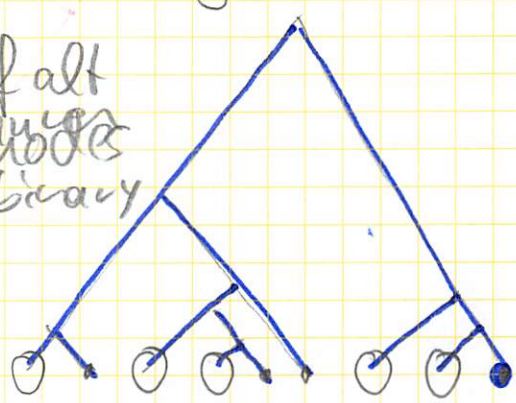
visited in correct order

alternating trees and binary trees

a visual bijection

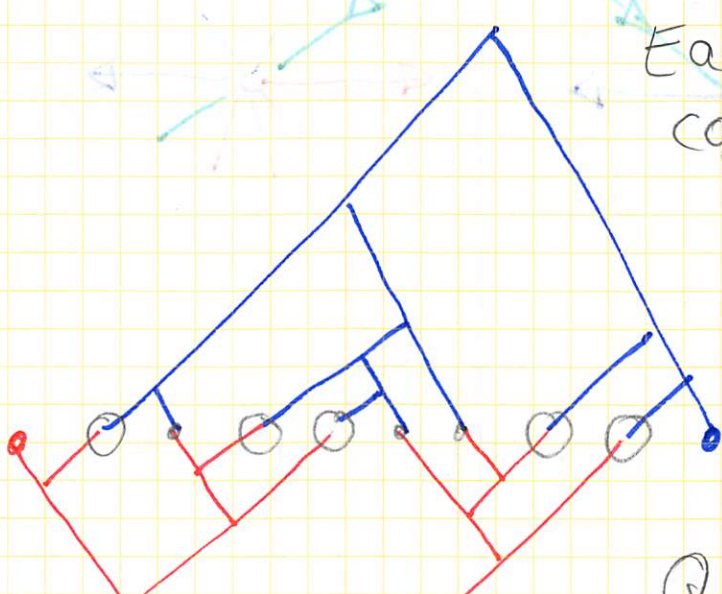


edges of alt tree map nodes of binary



The segment contact graphs.

Rückseite



Each sequent
corresponds
to a vertex
(contains)

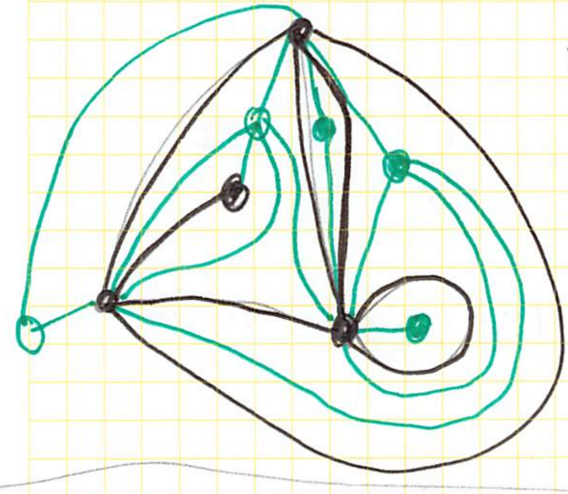
Each
edge of
Q (edge of
alt. trees)
yields a contact

Each contact an edge

Now we come to the theorem announced at the beginning

Angle graphs

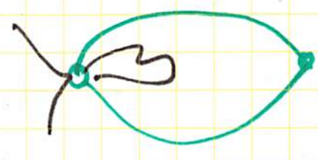
M a plane map the angle graph A_M has $V_A = V_M \cup F_M$ and an edge $(v\,f)$ for an angle where f meets v.



Every face of the angle graph corresponds to an edge and is a quadrangle

- Loops and Leaves

- cut vertices



multiedges

Assuming that M is 2 connected

A_M is a quadrangulation and $A_M = \text{cover}(P_{VF})$

A_M has segment contact representation

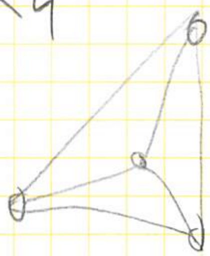
- extend edges
- add private stub to each segment

$\Rightarrow$ segments can be identifyed with elements of $sp(P_{VF})$

GIG repr. $\Rightarrow$ $\dim(sp P_{VF}) \le 4$

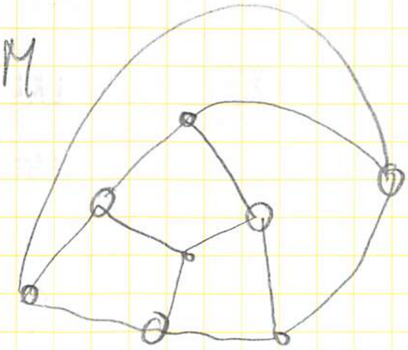
Example

$M = K_4$

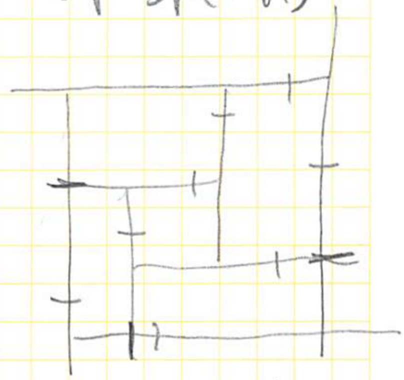


cube

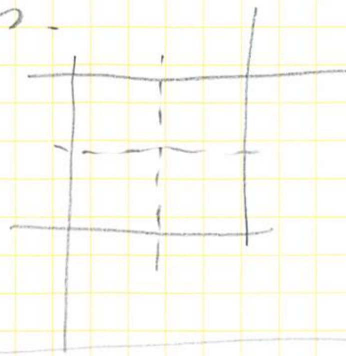
A_M



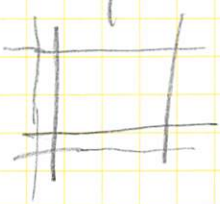
G|G repres of $sp(A_M)$



adding the edges.

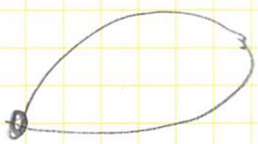


edge of outer quadr.



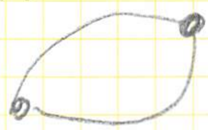
Dealing with loops:

M



subdivide

M'

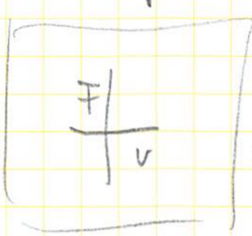


A_M is induced subgraph of $A_{M'}$

Dealing with cutvertices

Break – do each piece independently

outer piece



there is a corner

that can accommodate the inner piece

inner piece

make extremal

A

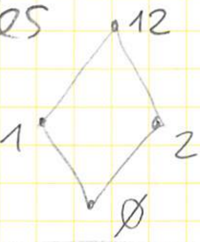
4.5 Incidence posets of complete graphs

Have seen: $\dim(K_n) \geq \log\log n$

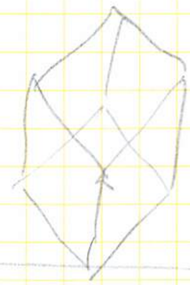
Today: A precise result – Hoşten Morris 1999

Def: Antichain A in $\mathcal{B}_t$ is HM if
$\forall S,T \in A \quad S \cup T \neq [t]$

Examples



$\emptyset$ $\{\emptyset\}$
$\{1\}$ $\{2\}$



$\emptyset$ $\{\emptyset\}$, $(1), (2), (3)$ (123)
$(1,2), (1,3)(2,3)$
$(12)(13)(23)$

THM $\boxed{\dim(K_n) = 1 + \min(t : \mathcal{B}_t \text{ has } \geq n \text{ HM antichains})}$

Rem: The HM-numbers are known up to $t=8$
see OEIS $\quad 2, 4, 12, 81, 2648$ $\quad \boxed{\dim K_{2648} = 6}$

$HM(t) \geq 2^{\binom{t}{\lfloor \frac{t-1}{2} \rfloor}} \qquad HM(t) \leq Ded(t) \leq 3^{\binom{t}{\lfloor \frac{t}{2} \rfloor}}$

This implies $\boxed{\dim(K_n) = t+1 \sim \log\log n + (\tfrac{1}{2} + o(1)) \log\log\log n}$

proof. Part 1: Lower bound

Let $\Sigma = \{ L_i : 0 \leq i \leq t \}$ a realizer
(each L_i a permutation)

We write $x <_i y$ if $x < y$ in L_i
we assume $L_0 = id_{[n]}$

B

Let $S_{xy} = \{ i : x <_i y \}$

and $A_x = \mathrm{Max}(S_{xy} : x <_0 y) \subseteq \mathcal{B}_{t-1}$

Prop $\mathcal{A} = \{ A_x : x \in [n] \}$ is a family of pw different HM antichains in $\mathcal{B}_{t-1}$

proof: $x <_0 y <_0 z$

different: Σ is a realizer $\Rightarrow \exists i$ with $x, z <_i y$

$\Rightarrow i \in S_{xy} \quad i \notin S_{yz} \Rightarrow S_{xy} \nsubseteq S_{yz}$

$\boxed{\text{Rück}}$ and $A_x \neq A_y$. ($A_n = \emptyset$ unique)

HM property $\exists j$ with $y z <_j x$

$\Rightarrow j \notin S_{xy} \quad j \notin S_{xz} \Rightarrow S_{xy} \cup S_{xz} \neq [t]$

$\Rightarrow A_x$ is HM $\square$

Part 2: upper bound

$\mathcal{B} = \mathcal{B}_t \qquad \mathcal{A}$ fam of HM antichains in $\mathcal{B}$

Fix a bij $[n] \longleftrightarrow \mathcal{A} \qquad x \longmapsto A_x$

Let λ be a linear extension of $\mathcal{B}$

We define: $v(A)$ the char. vector of A (components ordered according to λ)

with w_i the char vector of all sets not containing i

$$v_i(A) = v(A) +_2 w_i$$

Define $<_0 <_1 \cdots <_t$ on $[n]$ as follows

- $x <_0 y \Leftrightarrow v(A_x) <_{colex} v(A_y) \Leftrightarrow \underbrace{\max(A_x \Delta A_y)}_{M_{xy}} \in A_y$

Hier hatte ich einen kleinen Hänger

Der Ansatz:

Ang $A_x = A_y$

in A_x gibt es S mit $S_{xy} \subseteq S$

$\Rightarrow \exists z$ mit $S = S_{yz}$

nun gilt $S_{xy} \subseteq S_{yz}$

also kommt y nicht über x und z

C

- $x <_i y \iff v_i(A_x) <_{colex} v_i(A_y)$

$\iff i \in M_{xy} \in A_y$ or $i \notin M_{xy} \in A_x$

rewritten

$x <_i y$ and $x <_0 y \iff i \in M_{xy} \in A_y$

Claim. This is a realizer

Lex $x <_0 y <_0 z$

- in $<_0$ we have z over xy
- in $<_i$ we have x over yz

if $i \notin M_{xy} \in A_x$ and $i \notin M_{xz} \in A_x$

No such $i \Rightarrow M_{xy} \cup M_{xz} = [t]$

this contradicts the HM property of A_x

- in $<_i$ we have y over xz

if $i \in M_{xy} \in A_y$ $\quad i \notin M_{yz} \in A_y$

No such $i \Rightarrow M_{xy} \subseteq M_{yz}$

this contradicts the antichain property of A_y

$\square$

Dimension of 2 Levels of the Boolean Lattice

For levels $k < l < n$ in $\mathcal{B}_n$ we are interested in $\dim(k, l; n)$

We know $\dim(1, n-1; n) = n$ (standard ex)

$\dim(1, 2; n) \sim (1+o(1)) \log\log(n)$.

Remark 1. $\dim(i, j, k; n) = \dim(i, k; n)$

only min-max critical pairs

$\Rightarrow$ Remark 2. $\dim(1, k; n) \le \dim(1, k+1; n)$

D

Labad lower bound for $\dim(1,k;n)$

L can revert $\binom{n}{k+1}$ pairs (x,K)

There are $\binom{n}{k}(n-k)$ such pairs in $\mathcal{B}_n(1,k)$

$\Rightarrow \dim(1,k;n) \geq k+1$

More interesting results

Dushnik 1950

precise results for $\dim(1,k;n)$

for $2\sqrt{n}-2 \leq k \leq n-1$

A lower bound

Prop: $k \geq 2\sqrt{n}-2 \Rightarrow \dim(1,k;n) > n-\sqrt{n}$

Proof by monotonicity it is enough to prove it for $k = 2\sqrt{n}-2$

Suppose $L_1 \dots L_t$ is a realizer $t \leq n-\sqrt{n}$

Let T be the set of top elements $|[n]-T| \geq \sqrt{n}$

Let $S \subseteq [n]-T$ with $|S| = \sqrt{n}$ consider the restrictions $L_1' \dots L_t'$ of the L_i to S

$\exists x \in S$ which is top in $\leq \frac{t}{\sqrt{n}} \leq \frac{n-\sqrt{n}}{\sqrt{n}} = \sqrt{n}-1$ of the L_i'

Let $\Sigma_x = \{L_i' ; x \text{ is top in } L_i'\}$

and $T_x = \{y : y \text{ is top in } L_i \text{ and } L_i' \in \Sigma_x\}$

$K' = T_x \cup (S-x)$ $\quad |K'| \leq (\sqrt{n}-1)+(\sqrt{n}-1)$

$K \supseteq K'$ with $|K| = 2\sqrt{n}-2$ $\quad$ K never gets over x $\square$

$\uparrow$ not containing x